Teaching Ratio and Rates for Abstraction

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A group of practising teachers implemented the Teaching for Abstraction method for the Year 8 topic “Ratio and Rates”. The authors first constructed materials for a unit in which students explored familiar ratio and rates contexts, searched for similarities in their mathematical structure, defined the two concepts, and learned to apply these concepts to other contexts. After an introductory workshop, teachers taught the topic in six 1-hour lessons. They experienced considerable difficulties adapting the approach to the abilities and interests of their particular classes, but all students showed evidence of learning. It was concluded that, although Teaching for Abstraction shows promise, there are many factors that need to be taken into account if it is to be implemented in practice.

Teaching for Abstraction is an approach to teaching that takes account of the fact that most elementary mathematical ideas are abstractions from experience (Mitchelmore & White, 2004a). It consists of four steps, in which the teacher helps students to:

- *familiarise* themselves with the structure of a variety of relevant contexts,
- *recognise* the similarities between these different contexts,
- *reify* the similarities to form a general concept, and then
- *apply* the abstract concept to solve problems in related contexts.

The rationale for this approach is the theory of empirical abstraction (Mitchelmore & White, 2004b), where an abstract concept is seen to be “the end-product of ... an activity by which we become aware of similarities ... among our experiences” (Skemp, 1986, p. 21).

Teaching for Abstraction was originally developed from research on the learning of angle concepts in primary school (Mitchelmore & White, 2000). It has been applied successfully to the teaching of angle concepts in Stage 2 (NSW Department of Education and Training, 2003) and has also been trialed with decimals in Year 4 (Mitchelmore, 2002) and percentages in Year 6 (White & Mitchelmore, 2005). The project reported in this paper is one of two studies conducted in 2006 in which we continued to investigate student learning of multiplicative relations through Teaching for Abstraction.

Multiplicative Relations

A cursory look at the school mathematics curriculum shows that multiplicative relations underpin almost all number-related concepts studied in school (e.g., fractions, percentages, ratio, proportion, rates, similarity, trigonometry, and rates of change). Vergnaud (1983) called this set of concepts the multiplicative *conceptual field.* There is a long history of research showing that many children have considerable difficulty understanding these concepts (Behr, Harel, Post, & Lesh, 1992; Carpenter, Fennema, & Romberg, 1993; Harel & Confrey, 1994).

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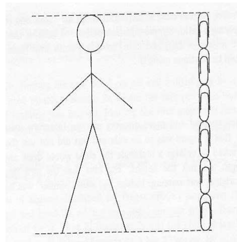
Ratio is a crucial multiplicative concept, and one of the most difficult ideas for students to come to terms with. Although working with ratios of the form *a:b* when *a* is a multiple of *b* causes few problems, cases where *a* is not a multiple of *b* can be highly problematic. Particularly prevalent is the so-called additive error (Misailidou & Williams, 2003) illustrated by the following, taken from a seminal study of a sample of 2257 English students aged 13-15 (Hart, 1982).

> You can see Mr Short's height measured with paperclips. When using matchsticks, Mr. Short's height is 4 match sticks. His friend Mr. Tall's height is 6 match sticks. How many paper clips are needed for Mr. Tall's height?



Mr Short

Only about one third of the students could correctly answer this question, with the majority opting for the answer of 8 (focusing on the additive difference 6 – 4 rather than the multiplicative 6:4).

An additional problem is the confusion that often arises between ratios and fractions. For example, if two boys and three girls sit at a table, the fractions $\frac{2}{5}$ and $\frac{3}{5}$ and the ratio 2:3 (often equated to $\frac{2}{3}$) arise. When a ratio connects two parts of the same whole, students may not adequately differentiate the part-part from the part-whole relationship (Clark, Berenson, & Cavey, 2003).

In New South Wales, the syllabus (NSW Board of Studies, 2002) suggests that rates be taught after ratios in Year 8 in a purely arithmetical context without any reference to slope or linear relations. It seems reasonable to expect the same errors as for ratio. The need to take account of different units may introduce additional errors. However, the fact that the two components clearly relate to different variables may reduce the prevalence of the additive error and eliminate the ratio-fraction confusion.

The Present Study

We hypothesise that students' poor performance on multiplicative tasks is at least in part due to the fact that curriculum materials rarely highlight their multiplicative nature. An emphasis on the underlying structure, including helping students to differentiate multiplicative from additive relations, could help students understand ratio and rates more deeply and enable the formation of stronger links to other multiplicative concepts. We propose Teaching for Abstraction as one way of focussing on this underlying structure.

A Teaching for Abstraction approach to ratios and rates would proceed as follows: Students would firstly explore various familiar situations involving ratios where they can solve simple problems without difficulty. They would then look for structural similarities between these calculations, explore the concepts involved, generalise and practise the procedure, and apply what they have learnt to new situations. This process would then be repeated for rates, emphasising the similarities and differences between rates and ratios.

This paper reports a research project designed to investigate whether it is possible for classroom teachers to implement the Teaching for Abstraction approach to ratio and rates in Year 8. A teaching unit was developed, teachers familiarised themselves with the approach and the content and then taught the unit, and we collected data on teacher and student learning. The study parallels a similar study of teaching percentages in Year 6 that is reported separately (White, Wilson, Faragher, & Mitchelmore, 2007).

Method

Participants

The students and teachers from six Year 8 classes in four schools participated. Two classes were from a selective boys' school, two were from comprehensive girls' schools and two were from comprehensive co-educational schools. Of the comprehensive classes, two contained high ability students, one average ability, and one low ability (as described by their teachers). In each class, five students were selected as a representative "target group".

Teaching Materials

Teachers were supplied with a unit consisting of six lessons, each intended to fit into a 60-minute period, covering the Ratio and Rates section of Outcome NS4.3 in the NSW Mathematics Syllabus (NSW Board of Studies, 2002). The materials included, alongside an orientation to Teaching for Abstraction, a suggested outline for each lesson together with black line masters that could be used for duplicating student worksheets. The six topics were as follows:

1. *Relative and absolute comparisons*
   Students explore a number of situations requiring the comparison between two values, and decide when it is more informative to compare them as they stand (absolutely) or in relation to each other or to other values (relatively).
2. *The concept of ratio*
   Students abstract the concept of ratio by looking for similarities between a variety of different situations where relative comparison is appropriate, and then explore its properties.
3. *Calculating with ratios*
   Students explore various methods of carrying out ratio calculations, including the unitary method, and are introduced to the concept of gradient.
4. *Fractions and ratios*
   A variety of practical situations is used to help students understand the similarities and differences between a ratio and a fraction.
5. *The concept of rate*
   Students explore a number of rate situations, and then explore the similarities and differences between rates and ratios.
6. *Calculating with rates*
   Students extend their skill at ratio calculations to similar calculations with rates, and explore the concept of speed.

Instruments

A short, task-based interview was used to assess students' understanding of the multiplicative structure of ratios and rates. It consisted of four questions focussed around four familiar multiplicative situations. Students were asked to perform various calculations and justify the methods they used. The content of the items is described in the Results section.

A 15-item unit quiz was constructed to assess students' calculation skills at the end of the unit. There were five items on simplification of decontextualised ratios, two on dividing in a given ratio, five on simplification of contextualised ratios, and three rates problems. Students were not asked to explain their answers because it was felt that deep understanding was better assessed through the interviews.

Procedure

The study took place in Term 3, 2006. In a one-day orientation workshop, teachers were introduced to Teaching for Abstraction and the proposed teaching unit.

They then taught the unit over a period of 2-3 weeks, and returned for a second workshop to share their experiences and assess the effectiveness of the unit.

The third author visited schools regularly during the teaching period. On her first visit, she interviewed all target students to assess their initial understanding of ratio and rates. On subsequent visits, she observed two lessons for each class and discussed each lesson with the teachers afterwards. On her final visit, she again interviewed the target students. Teachers also collected work samples from the target students in their class, and administered the unit quiz at the end of the teaching period.

The effectiveness of the teaching unit was assessed on the basis of the following data:

1. *Lesson evaluations* as shown by teachers' comments after each lesson and at the second workshop, the third author's observations, and student work samples;
2. *Student learning* as shown by the change in their understanding between the two interviews and their performance on the unit quiz.

Results

The topics taught in each lesson varied from school to school depending on the length of each period (varying from 40-80 minutes) and the ability level of the students. A further complicating factor was teacher unavailability: Four of the six classes were taught by at least one teacher who had not attended the orientation session. In two classes, the assigned teacher taught less than half the lessons.

The average- and low-ability students were only able to complete the first four lessons of the unit. These two classes also had one lesson in which only half the students were present, and there was no time to repeat the lesson. Students in the other four classes completed all the materials provided.

The results show that the students in the two selective schools performed at about the same level as the high-ability students in the two non-selective schools, so we have often pooled their data in the following.

Lesson Evaluations

Lesson 1 commenced with reports of a survey that students had been asked to administer, in which respondents were asked to indicate whether certain deductions from given data were valid. For example, given that "over the last 20 years in Australia, 10 people have died from crocodile bites and 12 people have died from dog bites", is it valid to deduce that being bitten by a dog has been more dangerous than being bitten by a crocodile? This was followed by discussions of the rationale for deciding The Biggest Loser (a well-known television program) and for assessing animal ages in human equivalents. Finally, the terms *relative* and *absolute* were defined and practiced.

These activities generated much heated discussion. Many students had enjoyed giving the survey to their parents and were amazed at the variety of responses. Most students seemed to understand that the survey data needed to be interpreted relatively, that percentage weight loss was the fairest criterion for The Biggest Loser, and that animal ages should be assessed relative to their average life span. However, many students were hindered by calculation difficulties – graphical displays sometimes helped. The teachers realised the activities were stimulating and felt that all students had understood the difference between relative and absolute comparisons. But they were clearly unused to leading discussions; two teachers found it difficult to curtail digressions peculiar to particular contexts and focus on the essential mathematical content.

In *Lesson 2*, students were asked to do some simple calculations involving "3 for the price of 2" sales, gear wheels, cinema queues on different nights, making playdough, scale drawings, and maps. It was expected students would solve these problems using their contextual knowledge. They were then asked to look for similarities between how they had solved each problem and to derive some generalisations. It was suggested that students use a bar model for making ratio comparisons. The concept of *equivalent ratios* was then introduced and practiced in a number of practical situations (sharing chocolate, making muffins, comparing fertilisers, and balancing voices in a choir).

Although there were a few context-related difficulties (especially with the map item), students seemed to be able to solve the given problems and recognise that they were each dealing with a relative comparison. Teachers said they would not normally have spent so much time on each context, but they seemed to be more familiar with these contexts and showed more skill in highlighting the underlying mathematical structure than in Lesson 1. The computation of equivalent ratios caused different problems for different students. The high-ability students recognised the similarity with equivalent fractions, but could not see how (for example) they could use a recipe for damper if they did not have a measuring cup to measure out the stated quantities. The students at the other end of the spectrum experienced mathematical difficulties (finding equivalent ratios) similar to those they had reportedly experienced with fractions.

Lesson 3 introduced the unitary method for solving proportion problems, and students applied it to some problems from the previous lesson. They then looked at the idea of gradient as a ratio and compared the gradients of some given slopes. The high-ability students enjoyed this lesson, but the other students again had difficulties calculating fractions and often confused the order of the two components of a ratio. The low-ability students attempted to work through all the examples but became confused and did not reach the intended outcomes. At this point, two of the three teachers of that class believed that the Teaching for Abstraction approach was not suitable for their students, so they decided to revert to their previous way of teaching the topic.

Lesson 4 was intended to address a problem, referred to in the introduction, that teachers had identified at the first teachers' meeting: the confusion between a ratio (relating two parts of a whole) and a fraction (relating a part to the whole). The two concepts were computed in a number of practical contexts and their different significance compared. The process of dividing a quantity in a given ratio was then addressed, after which the relation between ratio and percentage was explored.

The high-ability students had little difficulty with this lesson. One teacher supplemented the unit materials by beginning with a "drill and practice" exercise, but students did not make any errors on these calculations. The teacher of the low-ability class, who had reverted to the traditional approach, gave the students drill and practice after stating the rules to be followed. But students had difficulties both with the computations and with knowing which computations to do, and repeatedly questioned the rules they had been given. The teacher of the average-ability class used what seemed to be a more successful approach that certainly engaged the students. She worked only on the example with the smallest numbers. Students worked in pairs, and were required to explain their methods. The teacher then gave several slight variations before generalising and setting students a similar problem for homework. Unfortunately, this was one of the lessons for which only half the class was present.

Lesson 5 explored rates in several contexts (including run rates at cricket), encouraged generalisation by comparing rates and ratios, and addressed the issue of changing units. Gender differences appeared in relation to the cricket calculations and many students experienced difficulties converting units.

Lesson 6 gave more practice in rates, with a special emphasis on the rate-gradient relation in graphical representations (including distance-time graphs). The selective students seem to have covered all these topics previously. Some of the students had the same difficulties calculating with rates as they had experienced with ratios, especially when fractions or decimals were involved.

To summarise: Teachers and students liked having so many practical problems to discuss but were often distracted by contextual peculiarities. Teachers enjoyed "watching students think", and students enjoyed the challenge of making mathematical sense of interesting situations. The higher ability students had little difficulty abstracting the mathematical structure of ratios and rates, but the lower ability students were often hindered by difficulties manipulating fractions and decimals and often got frustrated. All the teachers agreed that they would be more selective of examples and teach the unit better next time.

Student Learning: Interview Results

Thirty students were interviewed before and after the unit had been taught. Figure 1 summarises the results.

Item 1, comparing the performance of basketball players who shot 20 goals from 40 shots or 25 goals from 50 shots, was answered well by all but two students before the unit was taught and by all students afterwards.

Item 2 posed three questions relating to mixing a given cordial. Only one student gave any additive answers (the same student before and after the unit). Among the others, the number of correct answers that were correctly explained increased from an average of 17 to 25.

Item 3 gave the positions of two runners at the start of a 100 m handicap race and 10 seconds into the race. Students were asked to predict the winner and the winning time. Only a few students from the lower ability classes showed any evidence of additive thinking, and the number of students giving correct responses increased from 17 to 22.

Item 4 asked students to suggest how the nutritional information on a food package could be used by people wishing to restrict their fat intake. The number of correct responses increased from 21 to 25. Interestingly, the number of students referring to the need to compare different foods decreased from 10 to 6, whereas the number stating that the information could be used to compare different serving sizes increased from 11 to 19.

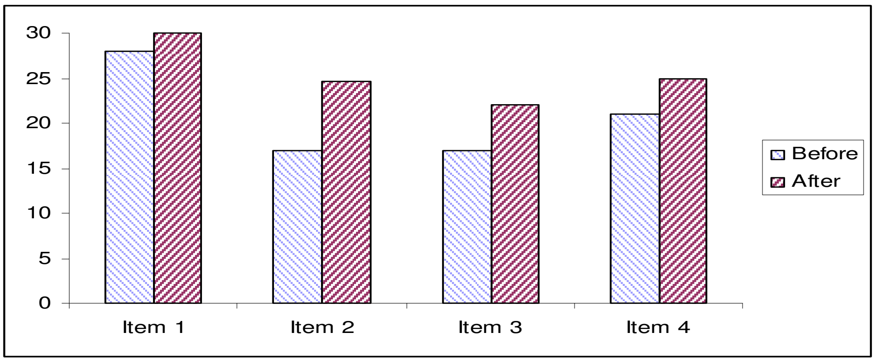


Figure 1. Number of correct interview responses before and after teaching.

To summarise: Most high-ability students had already learned to think multiplicatively before this unit was taught. During the course of the unit, some of the average- and low-ability students started to think multiplicatively and learnt how to perform the correct calculations.

Student Learning: Unit Quiz

Although there were no data available from a pre-test or from comparison classes, the 139 responses obtained to the unit quiz were still informative.

Students in the selective and top-stream classes performed at about the same rate (88% versus 83%), whereas students in the other two classes gave averages of 53% and 39% correct responses, respectively (partly because they did poorly on the rates questions, which they had not studied). The types of errors students made were also different in the three groups. In the high-ability classes, about 50% of the errors were related to units. Among the average-ability students, the most common error (30%) was incorrect multiplication or division. In low-ability students, the most common error (29%) was failing to reduce a ratio to its simplest form.

Discussion

We have learnt a great deal about the implementation of the Teaching for Abstraction method from this study. We discuss our findings under three headings: teaching, learning, and assessment.

Teaching

The teachers were all unfamiliar with the methodology of Teaching for Abstraction. In particular, they were not sure about when to let a discussion ramble, when to cut it off to draw out a mathematical point, and when to supply information or conventional terminology. Some teachers felt it was more difficult to maintain control when so many students wanted to talk at once. As a result, more time was taken than would normally have been available.

The contextualisation of the mathematics appeared to have been beneficial in arousing student interest, especially when teachers could bring in their own experiences (e.g., in raising rabbits). However, the converse also applied when teachers or students were unfamiliar with a context. For example, some teachers were not familiar with "The Greatest Loser" and some students were not interested in cricket, so these examples produced more mystification than enlightenment.

But the major difficulty that teachers experienced lay in adapting the given unit to the prior understanding of the students in their classes. In the higher ability classes, students were generally set to work through all the questions supplied after a minimal introduction, and teaching mainly resulted from discussion surrounding the more difficult questions. Some of these students were clearly frustrated at having to work through problems that did not challenge them. In the lower-ability classes, students could not cover all the material provided because of the calculation difficulties they experienced. Teachers had difficulty selecting exercises that would avoid these difficulties and still allow students to learn the concepts of ratio and rate.

Despite these difficulties, students generally seemed to enjoy the teaching approach and contributed willingly to the discussions. Teachers believed that, as a result, they came to know their students and appreciate their ways of thinking better. However, some students found it difficult to explain their thinking and others preferred working on their own, guided by answers at the back of the textbook.

Learning

There was some evidence of additive thinking in this study, although it never occurred among the high-ability students. Even students in the average- and lower-ability classes made relatively few errors due to additive thinking in the final interview and the quiz – most at least attempted to use multiplication or division. However, this may have been a result of the teaching unit's emphasis on multiplicative relations and may not represent any generalisable learning.

In the classes that had not studied rates, additive methods were more prevalent on the rates question in the quiz. Given a medicine label which says "Use 2 mL for each 5 kg body weight" and asked how much one should use for someone weighing 75 kg, one student proceeded to make a long table starting with 2 – 5, 3 – 6, 4 – 7, and ending with 75 – 78. He finally decided that you should use 72 mL for a 75 kg person. The same student used multiplication and division for all the questions on ratio. Without teaching, he clearly saw no connection between ratios and rates.

The major difficulty for the students in the higher-ability classes was in partitioning a given quantity in a given ratio. It appears that they often omitted the units because they believed ratios did not need units. There were also frequent errors in converting units.

Students in the average- and lower-ability classes had two main difficulties. Firstly, they often confused the ratio of two parts with the fraction for each part of the whole. This difficulty was known beforehand, but apparently Lesson 4 had not adequately addressed this misunderstanding – and student absence in the average-ability class only exacerbated the problem. Secondly, students often could not convert ratios to their simplest form because they were unable to recognise common factors. Converting ratios to unitary form was much easier because students could use their calculators for this. Unfortunately, no attempt was made to show students how to use the fraction mode on their calculators to reduce a ratio to simplest form.

All students, but particularly those in the lower ability classes, found the graphical representation of a ratio by a partitioned bar to be helpful. It would have been even more helpful had it previously been used in the teaching of fractions and percentages. Greater familiarity with the bar model could have enabled more students to relate the representation to the mathematical operations involved.

Assessment

In this study, we had to infer student understanding of ratios and rates mainly from lesson observations and teacher comments. Neither the interview nor the quiz adequately assessed non-routine learning (e.g., the connection with other topics) and what were judged as favourable responses could have been due to the influence of recent teaching. The absence of a pre-test on calculational skills was also a limitation on this study. Closer attention needs to be given to the assessment of multiplicative understanding in future studies.

Conclusions

This study has highlighted the difficulties in implementing Teaching for Abstraction in practice. Teachers were unfamiliar with the approach, and were unable to assimilate it in the one day briefing session. Furthermore, the frequent replacement of teachers during the course of the study meant that several classes were taught by teachers who had not been exposed to the philosophy behind the study unit at all. As a result, teachers were not in a position to do what they normally do as a matter of course: adapt the approach and the materials to the needs of the students in their class.

Encouragingly, all teachers said they would use at least some of the unit materials the following year, with modifications to suit their class. At that time, they would be more familiar with the approach, better able to choose appropriate contexts, and more confident about how to adapt the method to students' ability levels. Future implementation might be more successful if teachers, after an initial introduction to Teaching for Abstraction, were involved in the development of a revised ratio and rates unit. It may also be necessary to plan a general implementation of the model over a longer period of time, and not just for one unit. Professional development is obviously a key issue here.

Despite these difficulties, we still believe that Teaching for Abstraction holds promise. In this study, it appeared that many students were able to abstract the ratio concept from discussion of several contexts, and that more would have been able to do so if the contexts had been more appropriate for them. However it is clear that, in planning the practical implementation of the method, much more attention needs to be given to what needs to occur between the recognition of a concept and the application of that concept to new contexts – that is, the reification stage.

This study shows that pre-existing computational fluency plays an important role in the reification of ratio and rates concepts. Students who cannot recognise simple common factors or cannot perform simple multiplication and division calculations will have difficulties recognising multiplicative structure even in familiar contexts. Consequently, they will not be able to generalise across different contexts and abstract the desired concepts. Drill and practice exercises focussed on multiplication and division skills are unlikely to be helpful, and may only reinforce a feeling of failure. It is also likely to be unhelpful to restrict the examples to simple numbers that provide no challenge, because the multiplicative structure may then completely escape students' attention. More likely to be successful is careful grading in the difficulty of the arithmetical computations involved, more widespread use of graphical models, and the provision of electronic assistance once the underlying structure has been recognised.

The other side of the coin is that many Year 8 students may already have acquired the necessary computational fluency, even in ratio and rates problems. Instead of repeating unstimulating practice, such students would best deepen the reification of ratio and rates concepts by exploring the limitations of ratios and rates in practice as well as the links between them and other multiplicative concepts such as slope and enlargement.

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